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EQUIDISTANT CONIC MAP PROJECTIONS*

JOHN P. SNYDER

ABSTRACT. There are about a dozen published variations of equidistant conic map projections. They are identical when using the same actual standard parallels, but the parallels are chosen differently. Greater use of the spheroidal form of the equidistant conic with two standard parallels (especially when chosen by minimum-error procedures) is encouraged because of its true meridian scale and low-error compromise between Lambert's conformal conic and Albers' conical equal-area projections. Scale errors and coordinates for the three major conic projections and for their spherical versus spheroidal forms are compared in representative tables for U.S. maps. Relative scale errors using projections based on the sphere rather than the spheroid may also be calculated with relatively simple formulas.

THE most easily constructed map projections suitable for showing lands in the temperate zones are the conics. They have arcs of concentric circles for parallels of latitude and equally spaced straight radii of these circles for meridians of longitude. Angles between the meridians are smaller than the actual differences in longitude, the ratio being called n, the constant of the cone. If θ is the angle on the map between the central meridian of the projection and a longitude displaced by λ from the central meridian, θ equals n times λ.

The circular arcs of latitude may be spaced to obtain correct areas (equal-area or equivalent conics), correct local angles and shapes (conformal conics), or correct scale along the meridians (equidistant conics). While Albers' equal-area conic is often used for maps of the United States, and Lambert's conformal conic is used for maps of many countries, including the U.S., the equidistant conic is usually relegated to small areas. Actually it serves as a low-error compromise between the other two types, with the advantage of true linear scale along all meridians.

THE DE L'ISLE

The equidistant or simple conic originated in a rudimentary form with Claudius Ptolemy about 100 A.D. It was developed further by Johannes Ruysch in a world map of 1508, and by Gerardus Mercator in various maps of the mid-sixteenth century. Its present-day form, the equidistant conic with two standard parallels α_1 and α_2, along which all scales, angles, and small shapes are correct, is credited to De l'Isle (1745), a French map maker, who used it for a map of Russia.[1] In polar coordinates, the formulas may be written in direct form

$$\rho = R[(\alpha_2 \cos \alpha_1 - \alpha_1 \cos \alpha_2)/(\cos \alpha_1 - \cos \alpha_2) - \phi] \quad (1)$$

$$\theta = n\lambda \quad (2)$$

$$\text{where } n = (\cos \alpha_1 - \cos \alpha_2)/(\alpha_2 - \alpha_1). \quad (3)$$

R is the radius of the globe at the scale of the map, ρ is the radius of the circular arc for latitude ϕ, and other symbols are as above. All angles are given in radians throughout this paper unless the degree symbol (°) is used.

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[1] Johannes Keuning, "The History of Geographical Map Projections until 1600," *Imago Mundi,* Vol. 12 (1955), pp. 1–25, reference on p. 5; A. E. Nordenskiöld, *Facsimile-Atlas* (New York: Dover Publications, Inc., 1973. Reprint of 1889 publication), p. 86; and J. A. Steers, *An Introduction to the Study of Map Projections,* 15th ed. (London: Univ. of London Press, 1970), p. 239. It should be noted that Charles A. Schott, *A Comparison of the Relative Value of the Polyconic Projection Used on the Coast and Geodetic Survey, With Some Other Projections, Report of the Superintendent of the U.S. Coast and Geodetic Survey . . . June 1880* (Washington, D. C.: Government Printing Office, 1882), Appendix No. 15, p. 291, described a different De l'Isle equidistant conic with concentric parallels, but meridians which are not exactly radii of these parallels.

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A simpler form, with only one standard parallel ($\alpha_1 = \alpha_2 = \alpha$), may be constructed with the formulas $\rho = R\ (\cot \alpha + \alpha - \phi)$ and $\theta = \lambda \sin \alpha$, but it should not normally be used except for a very small country or subdivision. If, however, α is 90°, this projection becomes the well-known and useful polar form of the azimuthal equidistant projection: $\rho = R\ (\frac{1}{2}\pi - \phi)$, and $\theta = \lambda$.

It is also convenient to substitute a middle parallel $\phi_0' = \frac{1}{2}(\alpha_1 + \alpha_2)$, and $\delta' = \frac{1}{2}(\alpha_2 - \alpha_1)$, so that equations (1) and (3) become

$$\rho = R(\delta' \cot \delta' \cot \phi_0' + \phi_0' - \phi) \quad (4)$$

and

$$n = (\sin \delta' \sin \phi_0')/\delta'. \quad (5)$$

Equation (2) applies to all conics, with the appropriate n. Rectangular coordinates may be obtained by substituting into the following equations, which are also typical of conics:

$$x = \rho \sin n\lambda \quad (6)$$

$$y = \rho_0 - \rho \cos n\lambda, \quad (7)$$

where ρ_0 is the radius ρ calculated for an arbitrary latitude (such as ϕ_0') crossing the central meridian at the desired origin (x and $y = 0$).

The scale factor along the meridians is usually called h:

$$h = -\mathrm{d}\rho/R\mathrm{d}\phi = 1.0, \quad (8)$$

indicating true scale. The scale factor along the parallels is

$$k = \rho n/R \cos \phi = [(\alpha_2 - \phi) \cos \alpha_1 - (\alpha_1 - \phi) \cos \alpha_2]/(\alpha_2 - \alpha_1) \cos \phi = [\cos \phi_0' \cos \delta' + (\phi_0' - \phi) \sin \phi_0' (\sin \delta')/\delta']/\cos \phi. \quad (9)$$

The area scale factor is $A = hk = k$, while the maximum angular deformation ω at any point is the absolute value of 2 arc sin $[(1 - k)/(1 + k)]$.

The values of k are less than 1.0 between the standard parallels and greater than 1.0 beyond. At α_1 and α_2, k is 1.0. It does not vary with longitude; conics are especially suited for regions of predominant east-west extent.

OTHER VARIATIONS

There are many other published equidistant conics. It is interesting to note that all of them, if true to scale on the meridians, are completely identical to the De l'Isle, provided the same actual standard parallels α_1 and α_2 are used. The differences are in the means by which these parallels are chosen for the given region to be mapped, to give low error according to various criteria. To map a given region, standard parallels should be selected to minimize variation in scale, but many published alternatives are available. De l'Isle placed α_1 and α_2 at 1/4 and 3/4 of the distance between the upper and lower latitudes of the map. Deetz and Adams recommended spacings of about 1/6 and 5/6.[2] Hinks suggested 1/7 and 6/7.[3]

V. V. Kavraisky, for his fourth projection (1934), devised a significant procedure for determining the standard parallels of the equidistant conic by determining the "least mean square linear distortion within the territory to be represented." Each small interval of latitude, with its scale errors, is weighted in proportion to the area of the land or country actually embraced by it. Maling presented the derivation in his landmark article first describing most Russian projections for Western readers.[4] The resulting formulas may be written as follows:

$$n = \frac{(\Sigma Pmg)(\Sigma Pg) - (\Sigma Pg^2)(\Sigma Pm)}{(\Sigma Pm^2)(\Sigma Pg^2) - (\Sigma Pmg)^2} \quad (10)$$

$$C = \frac{(\Sigma Pm^2)(\Sigma Pg) - (\Sigma Pmg)(\Sigma Pm)}{(\Sigma Pmg)(\Sigma Pg) - (\Sigma Pg^2)(\Sigma Pm)} \quad (11)$$

where

n = the cone constant,

C = the radius of the equatorial parallel on the map for a sphere of radius 1.0,

P = the surface area of the region being mapped within a given zone of latitude, say 1°,

$m = \phi/\cos \phi$, for the center of each zone of latitude,

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[2] Charles H. Deetz and Oscar S. Adams, *Elements of Map Projection with Applications to Map and Chart Construction, U.S. Coast and Geodetic Survey, Special Publication No. 68*, 4th ed. (Washington, D.C.: Government Printing Office, 1934), p. 79.

[3] Arthur R. Hinks, *Map Projections* (Cambridge, England: Cambridge Univ. Press, 1912), p. 87.

[4] D. H. Maling, "A Review of Some Russian Map Projections," *Empire Survey Review*, Vol. 15 (1960), pp. 203–15, 255–66, and 294–303; reference on pp. 263–65.

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$g = 1/\cos \phi$, for the center of each zone of latitude,

and

Σ = the sum of values for each zone on the map, from minimum to maximum latitudes.

This writer has found it convenient to calculate these summations using Simpson's rule for numerical integration, providing increased accuracy.

After constants n and C are determined, α_1 and α_2 can be calculated by trial and error as the two values of α satisfying an adaptation of equation (9) with $k = 1.0$:

$$\cos \alpha + n\alpha = nC. \tag{12}$$

Kavraisky then constructed his equidistant conic according to the De l'Isle formulas from rounded-off values of the decimal angles so determined.

Several other equidistant conic formulas result from scale relationships chosen for the limiting northern and southern parallels of the map, ϕ_2 and ϕ_1, respectively.

Patrick Murdoch, a Scottish clergyman, presented two projections of this type in 1758, both of which maintain the correct total area of the map between parallels ϕ_1 and ϕ_2 and between any two meridians. The areas are not correct for each small portion, however.[5]
Murdoch I:

$$\rho = R[(\cot \phi_0 \sin \delta)/\delta + \phi_0 - \phi] \tag{13}$$

$$n = \sin \phi_0, \tag{14}$$

Murdoch III:

$$\rho = R(\delta \cot \delta \cot \phi_0 + \phi_0 - \phi) \tag{15}$$

$$n = (\sin \phi_0 \sin \delta \tan \delta)/\delta^2, \tag{16}$$

where

$$\phi_0 = \frac{1}{2}(\phi_1 + \phi_2)$$

$$\delta = \frac{1}{2}(\phi_2 - \phi_1).$$

The scale error $(k - 1)$ for III at ϕ_0 is a little less than half the identical errors $[(\tan \delta)/\delta]$ at ϕ_1 and ϕ_2, and of opposite sign. Each has two standard parallels which can be calculated by trial and error from equation (12), with $C = \rho$ for $R = 1.0$ and $\phi = 0$. For Murdoch III, for example, they are the two latitudes satisfying the equation:

$$\cos \alpha + n\alpha = (\sin \delta \cos \phi_0)/\delta + n\phi_0. \tag{17}$$

Actually, Murdoch thought that n for his third projection is the same as that for the first. Young reported the error in 1920 and corrected it as above.[6] Maurer applied this correction for n to the scale h along the meridians, instead, but this makes the projection into an "approximately equidistant" conic, with h a constant slightly greater than 1.0.[7]

Leonhard Euler, a Swiss mathematician in Russia, developed in 1777 an equidistant conic in which the absolute errors $(\rho\theta - R\lambda \cos \phi)$ of scale along the extreme and central parallels are equal and of opposite sign:[8]

$$\rho = R(\tfrac{1}{2}\delta \cot \tfrac{1}{2}\delta \cot \phi_0 + \phi_0 - \phi) \tag{18}$$

$$n = (\sin \phi_0 \sin \delta)/\delta. \tag{19}$$

$$\cos \alpha + n\alpha = \frac{1}{2}\cos \phi_0 (1 + \cos \delta) + n\phi_0. \tag{20}$$

Close and Clarke in 1911 described an equidistant conic used by the British Ordnance with the scale errors at the extreme parallels equal and opposite in sign to the scale error of the intermediate parallel γ of maximum error.[9] The formula for ρ is the same as equation (15), but n is found as follows:

$$1/n = \frac{1}{2}[\delta(\cot \phi_0 \cot \delta + 1)/\cos(\phi_0 - \delta) + (\delta \cot \phi_0 \cot \delta + \phi_0 - \gamma)/\cos \delta]. \tag{21}$$

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[5] Steers, op. cit., footnote 1, pp. 111–15. Original reference: Rev. Patrick Murdoch, "Of the Best Form of Geographical Maps," *Philosophical Transactions of the Royal Society*, Vol. 50 (II) (1758), p. 553 (not seen by author).

[6] Alfred Ernest Young, *Some Investigations in the Theory of Map Projections* (London: Royal Geographical Society, 1920), p. 40. See also A. R. Hinks, "Murdoch's Third Projection," *Geographical Journal*, Vol. 97 (1941), pp. 358–62.

[7] Hans Maurer, "Ebene Kugelbilder" (Ergänzungsheft Nr. 221 zu *Petermanns Geographische Mitteilungen*, Gotha, Germany, 1935). Translated, with foreword and editing by William Warntz, Harvard Papers in Theoretical Geography, 1968, item S50.

[8] Hinks, op. cit., footnote 3, pp. 82–84.

[9] Charles Frederick Close and Alexander Ross Clarke, "Map Projections," *Encyclopaedia Britannica*, 11th ed., Vol. 17 (1911), pp. 653–63; reference on p. 657.

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Latitude (γ) is calculated by trial and error from

$$\cot\gamma + \gamma = \delta \cot\phi_0 \cot\delta + \phi_0. \quad (22)$$

The standard parallels are the two values of α satisfying the equation

$$\cos\alpha + n\alpha = n(\delta \cot\delta \cot\phi_0 + \phi_0). \quad (23)$$

J. D. Everett independently developed a projection identical to Murdoch III, as corrected by Young. He received credit from Close and Clarke, but Murdoch was not mentioned.[10]

Four other Russian equidistant conics were developed during the twentieth century. Dmitri I. Mendeleev, the renowned chemist who developed the periodic table of elements in 1869, described a projection in 1907 which is actually the same as the De l'Isle, except that the (north) pole is one of the two standard parallels. (The German Wilhelm Schjerning's first projection, in 1882, is like Mendeleev's with the second standard parallel at 18° 36′ S. Lat., giving $n = \frac{1}{2}$.[11])

V. V. Vitkovsky presented his first projection in 1907 with the scale errors $(k - 1)$ along the extreme and central parallels equal and of opposite sign. The formula for ρ is again the same as equation (15), while n is

$$n = 2\tan\frac{1}{2}\delta(\sin\phi_0)/\delta. \quad (24)$$

Kavraisky's second projection (1934) has the scales k along the limiting parallels equal to each other and to the reciprocal of the scale along γ. The formulas are equations (15) and (25):

$$n^2 = \sin\phi_0 \sin\gamma\,(\sin\delta)/\delta, \quad (25)$$

where γ is calculated as in (22). Mikhailov is also credited with an equidistant conic similar in scale characteristics to Kavraisky II, but this writer has been unable to correct an apparent error in the equations presented.[12] Maling also listed two "approximately equidistant" conics by F. N. Krasovsky, his second (1922) and third (1925), each preserving the total area between limiting parallels and minimizing overall scale error in various ways, but having for h a constant slightly different from 1.0.[13]

As an example of the interrelation between these various equidistant conics, if a map of the United States is to range between latitudes $\phi_1 = 25°$ and $\phi_2 = 49°$ N, and any pair of meridians, the use of Murdoch III will lead to the following:

$$\phi_0 = \frac{1}{2}(25 + 49) = 37°; \quad \delta = \frac{1}{2}(49 - 25) = 12°.$$

From equation (16), n is found to be 0.60632. Equation (17), converted to degrees, becomes

$$\cos\alpha + 0.0105822\,\alpha° = 1.18435.$$

By trial and error, using more decimal places, α is calculated as 30.47965° and 44.38246°, which are α_1 and α_2, respectively. If these values are inserted into De l'Isle equations (1)–(3), or (2), (4), and (5), exactly the same coordinates will result as if Murdoch III equations (2), (15), and (16) are used for ϕ_1 and ϕ_2:

$$\rho = R\,(1.95336 - \phi)$$
$$\theta = 0.60632\,\lambda.$$

The scale factor k varies from about 0.99 to 1.015 for any of the foregoing equidistant conics with the same latitude limits (Fig. 1).[14]

Conversely, equations (16) and (17) may be rearranged to find the δ satisfying a given pair of standard parallels and an arbitrary ϕ_0 about halfway between α_1 and α_2:

$$\delta^2/\sin\delta\tan\delta = (\alpha_2 - \alpha_1)\sin\phi_0/(\cos\alpha_1 - \cos\alpha_2) \quad (26)$$

The solution to this also requires trial and error. If α_1 is 29½°, α_2 is 45½° and ϕ_0 is 37½°, there is no solution. If ϕ_0 is reduced to 37°, δ is 12.595° for a Murdoch III identical to a De l'Isle with α's of 29½° and 45½°, and extending from 24.4° to 49.6° latitude.

APPLICATION TO THE SPHEROID

It is now almost as fast to calculate coordinates of a conic projection with spheroidal (ellipsoidal) corrections as it is to use formulas

[10] Close and Clarke, op. cit., footnote 9, p. 658. See also Hinks, op. cit., footnote 6, p. 359.

[11] Maurer, op. cit., footnote 7, item S43.

[12] Maling, op. cit., footnote 4, pp. 256 (facing), 260.

[13] Maling, op. cit., footnote 4, pp. 261–62, and 266.

[14] Modeled after graph in Maling, op. cit., footnote 4, p. 258, but redrawn for different conditions and additional projections.

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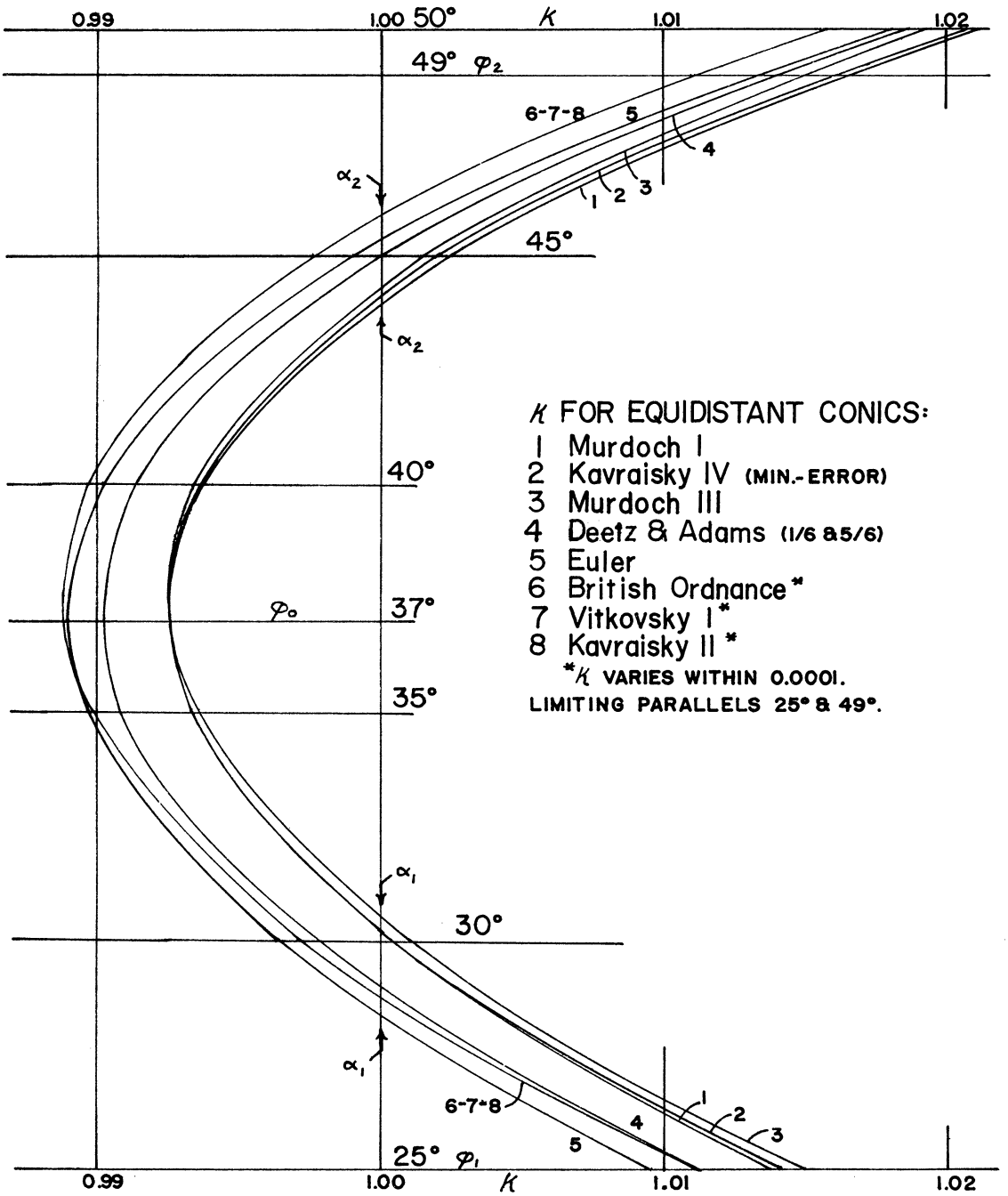

FIG. 1. Variation in scale factor k along parallels of various equidistant conic projections for map of U.S. with limiting parallels 25° and 49°.

based on the sphere, even with a programmable hand calculator.[15] Since the De l'Isle is an important alternative to the Albers and the Lambert, the relatively straightforward derivation

[15] Spheroidal rectangular coordinates have been obtained for the Lambert, Albers, and equidistant conic with, for example, the 100-step algebraic Texas Instruments SR-56, including automatic incrementing of latitude and longitude. This calculator is also suitable for the Kavraisky and other least-squares calculations.

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of the spheroidal formulas is given here. In spite of their value, the formulas do not seem to be included in available map projection refences.[16] Beginning with the well-known formulas for the lengths of arcs of the spheroidal meridians and parallels, the length of one radian of longitude along a parallel is often called N, where

$$N = a \cos \phi / (1 - e^2 \sin^2 \phi)^{1/2}, \quad (27)$$

while S will be made the symbol for the length of a meridian between latitudes ϕ_1 and ϕ_2:

$$S = a \int_{\phi_1}^{\phi_2} M \, d\phi, \quad (28)$$

where

$$M = (1 - e^2)/(1 - e^2 \sin^2 \phi)^{3/2},$$

a = the equatorial semiaxis of the earth, and

e = the eccentricity of the spheroid.[17]

To make the scale correct along the two standard parallels of the equidistant conic, equations (9) and (27) may be adapted as follows:

$$n \rho_{\alpha_2} = a \cos \alpha_2 / (1 - e^2 \sin^2 \alpha_2)^{1/2} \quad (29)$$

$$n \rho_{\alpha_1} = a \cos \alpha_1 / (1 - e^2 \sin^2 \alpha_1)^{1/2}. \quad (30)$$

The length of the meridians must be correct between α_1 and α_2:

$$\rho_{\alpha_1} - \rho_{\alpha_2} = a \int_{\alpha_1}^{\alpha_2} M \, d\phi. \quad (31)$$

Solving equations (29)–(31) for n and ρ_{α_2},

$$n = [\cos \alpha_1 / (1 - e^2 \sin^2 \alpha_1)^{1/2} - \cos \alpha_2 / (1 - e^2 \sin^2 \alpha_2)^{1/2}] / \int_{\alpha_1}^{\alpha_2} M \, d\phi \quad (32)$$

$$\rho_{\alpha_2} = a \cos \alpha_2 / n (1 - e^2 \sin^2 \alpha_2)^{1/2} \quad (33)$$

Then

$$\rho = \rho_{\alpha_2} + a \int_{\phi}^{\alpha_2} M \, d\phi. \quad (34)$$

To reduce computer time, the integral should be converted to a series:

$$S = a(\phi_2 - \phi_1)(1 - e^2/4 - 3e^4/64 - 5e^6/256 - \ldots) - a(\sin 2\phi_2 - \sin 2\phi_1)(3e^2/8 + 3e^4/32 + 45e^6/1024 + \ldots) + a(\sin 4\phi_2 - \sin 4\phi_1) \cdot (15e^4/256 + 45e^6/1024 + \ldots). \quad (35)$$

Constants for the Clarke 1866 spheroid (a = 6378206.4 meters and e^2 = 0.00676866) may then be inserted into equation (35), while converting $(\phi_2 - \phi_1)$ from radians to degrees:

$$a \int_{\phi_1}^{\phi_2} M \, d\phi = 111132.09 \, (\phi_2^\circ - \phi_1^\circ) - 16217 \, (\sin 2\phi_2 - \sin 2\phi_1) + 17 \, (\sin 4\phi_2 - \sin 4\phi_1) \quad (36)$$

in meters. The integrals in equations (32) and (34) are the same as the integral of (36) except for the substitution of the appropriate limits in place of ϕ_1 or ϕ_2.

Equations (32), (33), (34), (2), and (36) may then be used together with (6) and (7) for the rectangular coordinates. (The Clarke spheroid is used here because it is the usual standard in North America. The 1967 I.U.G.G. values of a = 6378160 meters and e^2 = 0.00669454 may be used as readily.)

For the scale factors of the spheroidal form of the projection, equations (8) and (9) become the following:

$$h = 1.0 \quad (37)$$

$$k = n \rho (1 - e^2 \sin^2 \phi)^{1/2} / a \cos \phi. \quad (38)$$

COMPARISON OF PROJECTIONS: SPHERE VS. SPHEROID

Corrections for the spheroid may be applied in another way to determine fairly easily the true scale errors for any projection based on the sphere, once the formulas or values for h and k are determined. This approach is especially suitable for conic and cylindrical projections, in which the meridians and parallels intersect at right angles, and h and k are the maximum and minimum scale factors (in the same or reverse order). The differential length of a radian of latitude along a meridian on the sphere is R, and on the spheroid is the derivative of equation (28) with respect to ϕ, or aM. The ratio of this length for sphere/spheroid is

[16] Spheroidal corrections for the equidistant variant used by the British Ordnance are mentioned in Close and Clarke, op. cit., footnote 9, p. 657, but no details are given.

[17] Deetz and Adams, op. cit., footnote 2, p. 94.

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TABLE 1.—SPHEROIDAL CORRECTION FACTORS TO APPLY TO SPHERICAL PROJECTIONS (CLARKE 1866 SPHEROID)

lat. ϕ	C_m[a]	C_p	C_s	C_a
90°	0.99548	0.99548	1.00000	0.99099
75	0.99617	0.99571	1.00046	0.99189
60	0.99803	0.99633	1.00170	0.99437
45	1.00058	0.99718	1.00341	0.99775
30	1.00313	0.99802	1.00511	1.00114
15	1.00499	0.99864	1.00636	1.00363
0	1.00568	0.99887	1.00681	1.00454
multiply by	h	k	h/k	hk

[a] $C_m = 1.0$ for 48.24° lat., and $C_a = 1.0$ for 35.32° lat.

$$C_m = R(1 - e^2 \sin^2 \phi)^{3/2}/a\,(1 - e^2). \quad (39)$$

Although R may theoretically be given any value from a to the polar semiaxis $a\,(1 - e^2)^{1/2}$, it should be intermediate and is often taken as the radius of a sphere with total surface area equal to that of the spheroid. This is the value used here:

$$R = a\{1/2 + [(1 - e^2)/4e] \ln [(1 + e)/(1 - e)]\}^{1/2} \quad (40)$$

$= 6370997$ meters for the Clarke spheroid.[18]

The length of a radian of longitude along a parallel on the sphere is $R \cos \phi$, and on the spheroid is found from equation (27). The ratio for sphere/spheroid is

$$C_p = R \cos \phi/[a \cos \phi/(1 - e^2 \sin^2 \phi)^{1/2}] = R\,(1 - e^2 \sin^2 \phi)^{1/2}/a. \quad (41)$$

The local shape correction ratio is

$$C_s = C_m/C_p = (1 - e^2 \sin^2 \phi)/(1 - e^2). \quad (42)$$

The area correction ratio is

$$C_a = C_m C_p = R^2(1 - e^2 \sin^2 \phi)^2/a^2(1 - e^2). \quad (43)$$

To determine actual scale errors of any given projection based on the sphere, relative to the spheroid, h may be multiplied by C_m, and k by C_p. For conics and cylindricals, the conformality factor h/k may be multiplied by C_s, and the area factor $A = hk$ by C_a (Table 1).

TABLE 2.—LAMBERT CONFORMAL CONIC PROJECTION (STANDARD PARALLELS 33° AND 45°)

Projection based on	sphere	sphere	spheroid
Scales relative to	sphere	spheroid	spheroid
lat. ϕ	*Polar coordinates:* ρ[a]		
50°	1.03373		1.03689
45	1.12154		1.12468
40	1.20852		1.21157
35	1.29536		1.29825
30	1.38274		1.38539
25	1.47127		1.47360
n	0.63048		0.63050
	Scale along meridian: h		
50	1.01394	1.01362	1.01389
45	1.00000	1.00058	1.00000
40	0.99464	0.99610	0.99467
35	0.99700	0.99932	0.99702
30	1.00665	1.00980	1.00662
25	1.02349	1.02744	1.02337
	Scale along parallel: k		
50	1.01394	1.01078	1.01389
45	1.00000	0.99718	1.00000
40	0.99464	0.99213	0.99467
35	0.99700	0.99477	0.99702
30	1.00665	1.00466	1.00662
25	1.02349	1.02172	1.02337
	Shape factor: h/k		
50	1.00000	1.00282	1.00000
45	1.00000	1.00341	1.00000
40	1.00000	1.00400	1.00000
35	1.00000	1.00457	1.00000
30	1.00000	1.00511	1.00000
25	1.00000	1.00560	1.00000
	Area scale: hk		
50	1.02807	1.02455	1.02798
45	1.00000	0.99775	1.00000
40	0.98932	0.98826	0.98936
35	0.99402	0.99409	0.99404
30	1.01335	1.01450	1.01328
25	1.04754	1.04975	1.04729

[a] for sphere, $R = 1$; for spheroid, $a = 6378206.4/6370997$ for equivalence to $R = 1$.

It is instructive to compare scale factors relative to sphere and spheroid for the three standard conic projections—Lambert, Albers and equidistant—using conventional standard parallels for U.S. maps of the first two and an approximately minimum-error pair for the third (Tables 2, 3, and 4). The formulas for the Lambert and Albers are not given here, but are available in U.S. Coast and Geodetic Survey publications.[19]

[18] Other reasonable values for R do not significantly change the concepts discussed. If the true meridian length is to be maintained, for example, R is 6367400 meters, from equation (35), only 0.056% less.

[19] Paul D. Thomas, *Conformal Projections in Geodesy and Cartography, U.S. Coast and Geodetic Survey Special Publication, No. 251* (Washington, D.C.: Government Printing Office, 1952), pp. 10–11. Oscar

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TABLE 3.—ALBERS' CONICAL EQUAL-AREA PROJECTION (STANDARD PARALLELS 29½° AND 45½°)

Projection based on / Scales relative to	sphere / sphere	sphere / spheroid	spheroid / spheroid
lat. ϕ	*Polar coordinates:* ρ^a		
50°	1.08475		1.08795
45	1.17142		1.17461
40	1.25921		1.26231
35	1.34730		1.35023
30	1.43503		1.43772
25	1.52186		1.52425
n	0.60284		0.60290
	Scale along meridian: h		
50	0.98296	0.98266	0.98303
45	1.00132	1.00190	1.00132
40	1.00915	1.01063	1.00911
35	1.00855	1.01089	1.00852
30	1.00108	1.00421	1.00107
25	0.98787	0.99168	0.98793
	Scale along parallel: k		
50	1.01733	1.01416	1.01727
45	0.99868	0.99586	0.99869
40	0.99093	0.98843	0.99097
35	0.99152	0.98929	0.99155
30	0.99892	0.99695	0.99893
25	1.01227	1.01052	1.01222
	Shape factor: h/k		
50	0.96622	0.96894	0.96634
45	1.00264	1.00606	1.00263
40	1.01839	1.02246	1.01831
35	1.01718	1.02183	1.01711
30	1.00216	1.00728	1.00215
25	0.97590	0.98136	0.97601
	Area scale: hk		
50	1.00000	0.99658	1.00000
45	1.00000	0.99775	1.00000
40	1.00000	0.99893	1.00000
35	1.00000	1.00007	1.00000
30	1.00000	1.00114	1.00000
25	1.00000	1.00211	1.00000

[a] for sphere, $R = 1$; for spheroid, $a = 6378206.4/6370997$ for equivalence to $R = 1$.

TABLE 4.—EQUIDISTANT CONIC PROJECTION (STANDARD PARALLELS 30° AND 44°)

Projection based on / Scales relative to	sphere / sphere	sphere / spheroid	spheroid / spheroid
lat. ϕ	*Polar coordinates:* ρ^a		
50°	1.09354		1.09681
45	1.18081		1.18406
40	1.26808		1.27124
35	1.35534		1.35834
30	1.44261		1.44537
25	1.52988		1.53233
n	0.60032		1.60036
	Scale along meridian: h		
50	1.00000	0.99969	1.00000
45	1.00000	1.00058	1.00000
40	1.00000	1.00146	1.00000
35	1.00000	1.00232	1.00000
30	1.00000	1.00313	1.00000
25	1.00000	1.00385	1.00000
	Scale along parallel: k		
50	1.02129	1.01811	1.02122
45	1.00248	0.99965	1.00247
40	0.99374	0.99123	0.99377
35	0.99327	0.99104	0.99330
30	1.00000	0.99802	1.00000
25	1.01336	1.01160	1.01329
	Shape factor: h/k		
50	0.97915	0.98191	0.97922
45	0.99752	1.00092	0.99753
40	1.00630	1.01032	1.00627
35	1.00678	1.01138	1.00675
30	1.00000	1.00511	1.00000
25	0.98682	0.99234	0.98688
	Area scale: hk		
50	1.02129	1.01780	1.02122
45	1.00248	1.00023	1.00247
40	0.99374	0.99268	0.99377
35	0.99327	0.99334	0.99330
30	1.00000	1.00114	1.00000
25	1.01336	1.01550	1.01329

[a] for sphere, $R = 1$; for spheroid, $a = 6378206.4/6370997$ for equivalence to $R = 1$.

In the area covered, the scale factors of the spheroidal projection relative to the spheroid and of the spherical projection relative to the sphere generally differ only in the fifth decimal place. If the spherical projection is related to the spheroid, the parallels of true scale shift slightly from the nominal standard parallels toward the extremities of the map. These shifted true-scale parallels are not free of all distortion, while the standard parallels on the spheroidal projection are. Though k is 1.0, h is slightly different. The tables delineate the well-known fact that the spherical form of Albers is not quite equal-area with reference to the spheroid, the spherical equidistant is not truly equidistant, and the spherical Lambert is not conformal.

The maximum linear scale error using a spherical projection rather than spheroidal for a map of the U.S. is seen to be about one-third of one percent for the three projections. Since the spheroidal projection in this range has a maximum linear error of a little over two percent, one-third of one percent is a small but

S. Adams, *Tables for Albers Projection, U.S. Coast and Geodetic Survey Special Publication, No. 130* (Washington, D.C.: Government Printing Office, 1927), pp. 4–6.

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TABLE 5.—EQUIDISTANT CONIC PROJECTION (STANDARD PARALLELS 37° AND 40°)

Projection based on / Scales relative to	sphere / sphere	sphere / spheroid	spheroid / spheroid
lat. ϕ	*Polar coordinates:* ρ^a		
41°	1.21325		1.21639
40	1.23071		1.23382
39	1.24816		1.25125
38	1.26561		1.26867
37	1.28307		1.28609
36	1.30052		1.30351
n	0.62244		0.62245
	Scale along meridian: h		
41	1.00000	1.00129	1.00000
40	1.00000	1.00146	1.00000
39	1.00000	1.00164	1.00000
38	1.00000	1.00181	1.00000
37	1.00000	1.00198	1.00000
36	1.00000	1.00215	1.00000
	Scale along parallel: k		
41	1.00062	0.99804	1.00062
40	1.00000	0.99747	1.00000
39	0.99969	0.99722	0.99970
38	0.99970	0.99729	0.99970
37	1.00000	0.99764	1.00000
36	1.00060	0.99830	1.00059
	Shape factor: h/k		
41	0.99938	1.00326	0.99938
40	1.00000	1.00400	1.00000
39	1.00031	1.00442	1.00030
38	1.00030	1.00454	1.00030
37	1.00000	1.00435	1.00000
36	0.99940	1.00386	0.99941
	Area scale: hk		
41	1.00062	0.99932	1.00062
40	1.00000	0.99893	1.00000
39	0.99969	0.99886	0.99970
38	0.99970	0.99909	0.99970
37	1.00000	0.99962	1.00000
36	1.00060	1.00044	1.00059

[a] for sphere, $R = 1$; for spheroid, $a = 6378206.4/6370997$ for equivalence to $R = 1$.

significant portion. From the polar coordinates given, it can be found that the meridians on the equidistant conic for the spheroid are 0.0008 radius shorter for the range given than those for the spherical projection. On a map at a scale of one to five million, this is only one millimeter, but worth correcting, especially since the spheroidal form of the projection is so easily computed with programmable calculators.

On a map of an average-sized state, such as Missouri, or a smaller area, the one-third-percent spheroidal correction far outweighs the maximum one-sixteenth-percent linear scale error with the equidistant conic if the spherical projection is used (Table 5). The spheroidal form of the projection is therefore essential to precision mapping of such an area. Of course, the spheroid is indeed the basis for topographic maps and State Plane Coordinate systems; other projections are involved, but they have almost negligibly different linear errors for the area involved.[20]

TABLE 6.—COMPARISON OF SCALE FACTORS FOR THREE PRINCIPAL CONICS (STANDARD PARALLELS 29½° AND 45½°) CLARKE 1866 SPHEROID

lat. ϕ	Lambert	Albers	Equidistant
	Scale along meridian: h		
50°	1.01522	0.98303	1.00000
45	0.99877	1.00132	1.00000
40	0.99114	1.00911	1.00000
35	0.99134	1.00852	1.00000
30	0.99885	1.00107	1.00000
25	1.01352	0.98793	1.00000
	Scale along parallel: k		
50	1.01522	1.01727	1.01622
45	0.99877	0.99869	0.99873
40	0.99114	0.99097	0.99105
35	0.99134	0.99155	0.99144
30	0.99885	0.99893	0.99889
25	1.01352	1.01222	1.01285
	Shape factor: h/k		
50	1.00000	0.96634	0.98404
45	1.00000	1.00263	1.00127
40	1.00000	1.01831	1.00904
35	1.00000	1.01711	1.00863
30	1.00000	1.00215	1.00111
25	1.00000	0.97601	0.98731
	Area scale: hk		
50	1.03068	1.00000	1.01622
45	0.99755	1.00000	0.99873
40	0.98235	1.00000	0.99105
35	0.98275	1.00000	0.99144
30	0.99771	1.00000	0.99889
25	1.02723	1.00000	1.01285

If the three spheroidal conics are compared, using the same standard parallels, the scale factors for the equidistant are seen to be nearly the mean of those for the other two (Table 6).

MINIMUM-ERROR EQUIDISTANT CONICS

Since the values of h and k are so nearly identical for the spherical and spheroidal forms of each of these conic projections, it is convenient to use the spherical form to determine

[20] Hugh C. Mitchell and Lansing G. Simmons, *The State Coordinate Systems (A Manual for Surveyors), U.S. Coast and Geodetic Survey Special Publication, No. 235* (Washington, D.C.: Government Printing Office, 1945).

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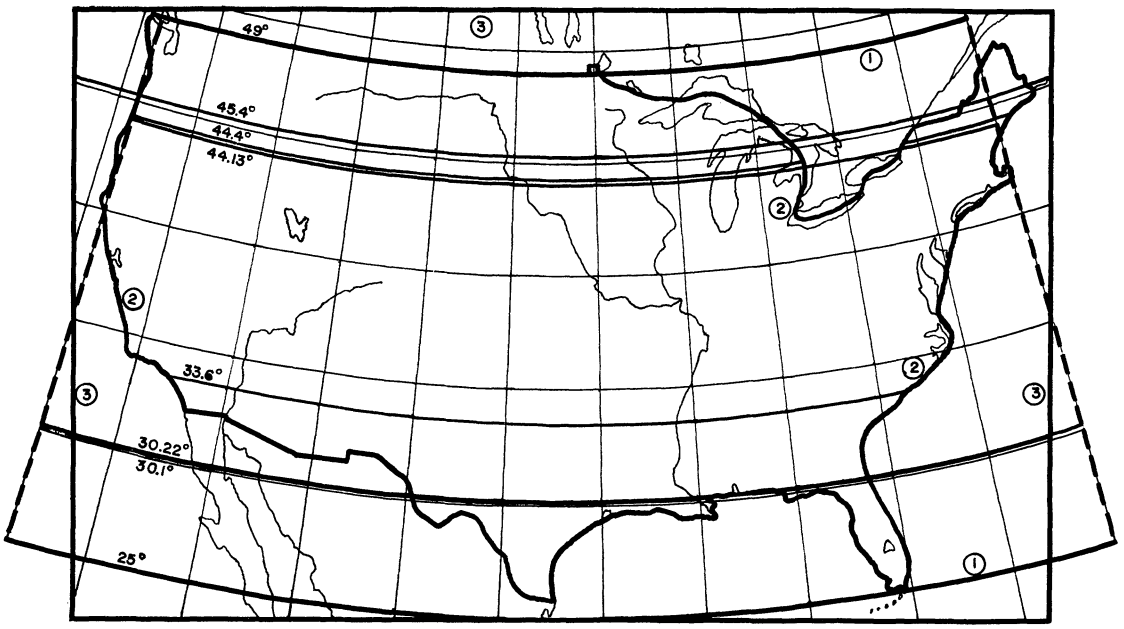

FIG. 2. Equidistant Conic Projection for map of the United States (standard parallels 30° and 44° N. Lat.), showing standard parallels for minimum error with varying boundary conditions, using Kavraisky's analysis.

which standard parallels will give the overall minimum scale errors for a given map, and then to use these standard parallels for the spheroidal projection to determine coordinates of an optimum map.[21] Such a calculation for optimum standard parallels is based on the classic "least squares" principle, that the sums of the squares of the scale errors be made a minimum.

There have been several analytical applications of this principle to various projections, and specifically to conic projections by N. J. Zinger in 1916, by Young in 1920, by Krasovsky in 1922 and 1925, and by Kavraisky in 1934.[22] Programmable calculators may be satisfactorily used, applying Simpson's rule for numerical integration to either Kavraisky's formulas or similar formulas for calculating minimum-error constants or mean errors for other conics. The following calculations are based on 1° steps of latitude.

The optimum standard parallels on an equidistant conic for a U.S. map will have various values depending on the boundary conditions shown. Each of these values can be determined with Kavraisky IV [equations (10)–(12)]. If the boundaries are the 25th and 49th parallels and any two meridians, the standard parallels for minimum error are 30.22° and 44.13°, giving a root-mean-square error $(k-1)$ of 0.00659 (region 1, Fig. 2). With these values rounded off to 30° and 44°, respectively, the resulting equidistant conic has a root mean square error of 0.00660 (Table 4). Murdoch I, with 30.18° and 44.03° as the actual standard parallels for these limits, has a root-mean-square error only 0.03% higher than minimum. Murdoch III, with 30.48° and 44.38°, is 0.3% above minimum. The Murdoch projections are very low-error, but not actually minimum-error, as is sometimes stated.

Young preferred Murdoch III to a minimum-error conic (with this type of boundaries) because Murdoch III's maximum error (1.49% at either 25° or 49° in this example) is less than that of the minimum-error projection (1.60% at 49° lat.), and the mean error is only slightly greater.[23] Murdoch I is better

[21] Close and Clarke, op. cit., footnote 9, p. 657, stated this conclusion.

[22] Maling, op. cit., footnote 4, pp. 263, 265, and 266. Young, op. cit., footnote 6, pp. 22–56. See also D. H. Maling, *Coordinate Systems and Map Projections* (London: George Philip and Son Ltd., 1973), pp. 70–71, 173–75, and 239.

[23] Young, op. cit., footnote 6, pp. 40–41. Actually Young was comparing Murdoch with a minimum-error conic which he developed, and which is almost but not exactly equidistant. Values of n and ρ differ

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than III at 25°, worse at 49°. The use of 29½° and 45½° for α_1 and α_2 results in a root-mean-square error of 0.00704, about 7% higher.

If, however, the map region is to be confined to the conterminous U.S. alone, without oceans, Canada, or Mexico, the calculated parallels are 33.6° and 44.4° (region 2, Fig. 2). If the boundaries are taken as the edges of a rectangle containing a map of the U.S. and adjacent lands and waters, 30.1° and 45.4° are calculated (region 3, Fig. 2). The optimum standard parallels for a Lambert conformal or Albers equal-area conic with minimum error lie within a degree of those for the equidistant with the same boundaries. Therefore, the 3½° difference between the southern standard parallels usually used for these projections can only be justified by different assumed limits to the U.S. map (Tables 2 and 3).[24]

The interest in the two-point equidistant projection developed by Close and Maurer, the azimuthal equidistant projection, and the Chamberlin Trimetric projection heavily used by the National Geographic Society indicates the importance given to distance preservation.[25] Tobler has recently developed an "Empirical Map Projection" for the United States, neither conic nor azimuthal, based on minimum errors in measuring 2080 distances between 65 intersections of latitudes and longitudes.[26]

With its errors presenting a balance between those of conformal and equal-area conics, the spheroidal form of the equidistant conic would seem to merit more consideration than it has received for accurate, easily drawn maps of regions the size of the U.S., as well as those smaller.

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from Kavraisky IV only in the fifth decimal place ($R = 1$) for the U.S. map limits discussed in this article.

[24] Formulas for Lambert and Albers minimum-error conics are similar to equations (10)–(12), but are derived from principles detailed by Maling, op. cit., footnote 4, pp. 263–65. The formulas may be the same as those of Zinger, but they are not actually given by Maling. For region 1 (Fig. 2) boundaries, Lambert standard parallels are 30.05° and 43.87°; Albers' are 30.39° and 44.38°. For region 2 boundaries, almost the same differences from the equidistant prevail. The relationship varies under some other conditions.

[25] Charles Close, "Note on a Doubly-Equidistant Projection," *Geographical Journal*, Vol. 57 (1921), pp. 446–48. idem., *Note on Two Double, or Two-point Map Projections*, (1922), 8 pp., reviewed in *Geographical Journal*, Vol. 60 (1922), pp. 297–99, by A. E. Young. Wellman Chamberlin (1908–1976) was NGS chief cartographer for many years.

[26] W. R. Tobler, "Numerical Approaches to Map Projections," *Sonderdruck aus Festschrift für Erik Arnberger*. (Vienna: Franz Deuticke, 1977), pp. 51–64. In English.

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